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Large Deviations and Ergodicity for Spin Particle Systems

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In this paper we investigate the large deviation principle (LDP) for spin particle systems with possibly vanishing flip rates. The situation turns out to be much more complicated if the flip rates are allowed to be zero than the one considered by Dai, where the systems are assumed to have strictly positive flip rates. The upper and lower large-deviation bounds are studied, respectively. The two governing rate functions are compared and a variational principle is given. We then apply the results to obtain some new large-deviation estimates for the occupation times of attractive systems. In particular, we prove a strong form of exponential convergence for ergodic systems.

KEY WORDS: Interacting particle system; large deviation principle, ergodicity.

1. INTRODUCTION

The large deviation principle (LDP) plays an important role in studying an interacting particle system. It gives estimates of the probabilities that the system has large fluctuations away from its stationary measures. The large deviation (LD) rate function can be used to characterize these measures if one obtain the corresponding variational principle. Recently, Dai Pra[4,5] developed an approach to study the LDP for the space-time empirical processes of spin particle systems. The results obtained apply to all spin systems with strictly positive and translation invariant flip rates having finite range interactions. A corresponding variational principle has also been proved. A certain kind of local space-time Gibbsian structure plays a key role in the study.

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In the present paper we are interested in spin systems with not necessarily strictly positive flip rates, including the well known contact processes and voter models. The situation turns out to be much more complicated due to the vanishing of the flip rates. This can be seen from at least two aspects. First, the asymptotic behaviours depend more heavily on the initial distributions. For example, if a system has some traps, then starting from any one of them, the LDP of the process is trivial or may fail. So in general, to obtain a nontrivial LDP, conditions on the initial distributions are needed and we should not expect to obtain a uniform LDP as in refs. 4 and 5. Secondly, it can be seen from our arguments that the effects of boundary conditions are not negligible. In this paper we mainly study systems with periodic boundary conditions. We find two proper rate functions H_0 and H^0 and show that H_0 governs the LD upper bounds for the system starting from any initial distribution. We also give a reasonable condition on the initial distributions so that we can obtain the LD lower bounds governed by H^0. The two rate functions are compared. Though we have not shown that $H_0 = H^0$, we get that $H_0 \leqslant H^0$ and they coincide at least on $\{H^0 < \infty\}$. Furthermore, by showing a variational principle we see that both H_0 and H^0 can be zero only at the stationary measures of the system, so the probabilities of large fluctuations the system having away from these measures will decay exponentially fast as time tends to infinite. These results are then applied to attractive systems. We obtain some new large deviation estimates for their occupation times. In particular, we show a strong form of exponential convergence for ergodic systems.

For a spin system we refer to a continuous time Feller Markov process with state space $E = \{-1, 1\}^{\mathbf{Z}^d}$ $(d \geqslant 1)$. For $\eta \in E$, $\eta(i)$ is interpreted as the spin at site $i \in \mathbf{Z}^d$. The evolution of the system in time is characterized by a family of flip rates $\{c(i, \eta), i \in \mathbf{Z}^d, \eta \in E\}$, where for each i, $c(i, \cdot)$ is a nonnegative continuous function on E. The system changes its state at site i with probability rate $c(i, \cdot)$. So we define the following generator:

$$L_c f(\eta) = \sum_{i \in \mathbf{Z}^d} c(i, \eta)[f(\eta^i) - f(\eta)], \qquad \eta \in E \tag{1.1}$$

where f is a cylindric function on E, i.e., $f(\eta)$ depends on η only through the coordinates of η in some finite subset of $\mathbf{Z}^d$. For $i \in \mathbf{Z}^d$ and $\eta \in E$, $\eta^i \in E$ is defined by $\eta^i(j) = \eta(j)$ if $j \neq i$, $= -\eta(i)$ if $j = i$. Throughout this paper we assume that the flip rates c are translation invariant with finite range interactions, i.e., there exist a finite subset U_0 of $\mathbf{Z}^d$ and a nonnegative function c_0 on E which is not identically zero, such that $c_0(\eta)$ depends on η only through the coordinates of η in U_0 and

$$c(i, \eta) = c_0(\theta_i \eta), \qquad \forall i \in \mathbf{Z}^d, \quad \eta \in E$$

where θ_i is the shift operator on E defined by $(\theta_i\eta)(j)=\eta(j+i)$, $j\in\mathbf{Z}^d$. Under this assumption, there exists a unique Feller Markov process $\{P^0_\eta, \eta\in E\}$ on $(\Omega, \mathscr{B})$ with the closure of L_c as its infinitesimal generator (cf. ref. 11), where $\Omega = D(\mathbf{R}, E)$ is the space of E valued right continuous with left limit functions on $\mathbf{R}$, endowed with the Skorohod topology, $\mathscr{B}$ is the Borel σ-field.

We then define the family of systems with periodic boundary conditions. For $n\geqslant 1$, let $\Lambda_n=\{1,..., n\}^d$, $E_n=\{-1, 1\}^{\Lambda_n}$ and define

$$L^n_c f(\eta)=\sum_{i\in\Lambda_n} c(i, \eta^{(n)})[f(\eta^i)-f(\eta)], \qquad \eta\in E \tag{1.2}$$

where $\eta^{(n)}$ is the n-periodic element of η defined by

$$\eta^{(n)}(i+kn)=\eta(i), \qquad \forall i\in\Lambda_n, \quad k\in\mathbf{Z}^d$$

with $kn=(k_1 n,..., k_d n)$. The Markov process determined by L^n_c is denoted by $\{P^{0,n}_\eta, \eta\in E\}$, which is in fact a Markov process on $(\Omega_n, \mathscr{B}_n)$, where $\Omega_n=D(\mathbf{R}, E_n)$, $\mathscr{B}_n$ is the Borel σ-field. Now we define the space-time empirical processes on $(\Omega, \mathscr{B})$. For $t\in\mathbf{R}$ and $i\in\mathbf{Z}^d$, let $\theta_{t,i}$ be the shift operator on Ω given by

$$(\theta_{t,i}\omega)_s(j)=\omega_{s+t}(i+j), \qquad s\in\mathbf{R}, \quad j\in\mathbf{Z}^d$$

For $n\geqslant 1$ and $\omega\in\Omega$, the space-time periodic element ω^n is defined by

$$\omega^n_{t+ln}(i+kn)=\omega_t(i) \qquad \text{for} \quad 0\leqslant t<n, \ l\in\mathbf{Z}, \ i\in\Lambda_n, \ k\in\mathbf{Z}^d$$

Now define a family of probability measures on $(\Omega, \mathscr{B})$ as follows:

$$R_{n,\omega}(A)=\frac{1}{n^{d+1}}\sum_{i\in\Lambda_n}\int_0^n \delta_{\theta_{t,i}\omega^n}(A)\,dt, \qquad n\geqslant 1, \ \omega\in\Omega, \ A\in\mathscr{B}$$

where $\delta_\omega(A)=1$ if $\omega\in A$; $=0$, otherwise. It is clear that $R_{n,\omega}\in M_s(\Omega)$, the space of all $\{\theta_{t,i}, t\in\mathbf{R}, i\in\mathbf{Z}^d\}$-invariant probability measures on $(\Omega, \mathscr{B})$, provided with the weak topology. One of the main objects of this paper is to study the LDP of $\{P^{0,n}_\eta(R_n\in\cdot), n\geqslant 1)\}$ and $\{P^0_\eta(R_n\in\cdot), n\geqslant 1\}$. To this end, as in refs. 4 and 5, we will use a noninteracting system as the reference family. Denote by $\{P_\eta, \eta\in E\}$ the Markov process with $c(i,\eta)\equiv 1$. Then it is shown in refs. 4 and 5 that there exists a nonnegative function H on $M_s(\Omega)$ with compact level sets (i.e., $\forall a\geqslant 0$, $\{H(Q)\leqslant a\}$ is a compact subset of $M_s(\Omega)$), such that for each $A\in\mathscr{B}$,

$$-\inf_{Q\in A^0} H(Q) \leqslant \liminf_{n\to\infty} \frac{1}{n^{d+1}} \log \inf_{\eta\in E} P_\eta(R_n\in A)$$

$$\leqslant \limsup_{n\to\infty} \frac{1}{n^{d+1}} \log \sup_{\eta\in E} P_\eta(R_n\in A)$$

$$\leqslant -\inf_{Q\in \bar{A}} H(Q) \tag{1.3}$$

where A^0 and $\bar{A}$ are the interior and closure of A respectively. H is called the rate function.

Now we define the two rate functions we need. For $\omega\in\Omega$, let

$$0\equiv\tau_0(i)<\tau_1(i)<\cdots<\tau_k(i)<\cdots$$

be the successive jump times of $\omega_t(i)$ in $[0,\infty)$, and set

$$N_i(t)=N_i(t,\omega)=\sum_{k=0}^{\infty} I_{\{\tau_k(i)\leqslant t\}},\qquad i\in\mathbf{Z}^d,\quad t\geqslant 0$$

which is nondecreasing and right continuous in t. Then define for $Q\in M_s(\Omega)$

$$H_0(Q)=\begin{cases} H(Q)-E^Q[1-c_0(\omega_0)+\int_0^1 \log c_0(\omega_{t^-})\,N_0(dt)], \\ \qquad\qquad \text{if}\quad H(Q)<\infty, \\ +\infty \qquad \text{if}\quad H(Q)=\infty \end{cases}$$

with the convention that $-\infty\cdot 0=0$, where $\omega_{t^-}=\lim_{s\to t^-}\omega_s$. Noticing that $\log c_0(\eta)$ is bounded above, we see that H_0 is well defined. It will govern the LD upper bounds.

Next we define the rate function to be used to govern the LD lower bounds. First define for $n\geqslant 1$

$$F_n=\sigma(\omega_t(i)\colon 0\leqslant t\leqslant n,\, i\in\Lambda_n)$$

For $Q\in M_s(\Omega)$, denote by Q_ω^p the regular conditional probability distribution (r.c.p.d.) of Q given the σ-field $\sigma(\omega_t(i)\colon t\leqslant 0)$. Then define

$$H_n(Q)=E^Q\log\left(\left.\frac{dQ_\omega^p}{dP_{\omega_0}^{0,n}}\right|_{F_n}\right)$$

and

$$H^0(Q) = \begin{cases} \limsup\limits_{n\to\infty} \dfrac{1}{n^{d+1}} H_n(Q), & \text{if } H(Q) < \infty, \\ +\infty, & \text{if } H(Q) = \infty \end{cases}$$

Now we can state our first theorem concerning the two governing functions.

Theorem 1. Let H_0 and H^0 be defined as above. Then we have the following conclusions.

(1) If $H^0(Q) < \infty$, then $H_0(Q) = H^0(Q)$. Hence $H_0 \leqslant H^0$.

(2) H_0 is lower semicontinuous (l.s.c.) with compact level sets.

(3) If $H_0(Q) = 0$, then $Q^p_\omega = P^0_{\omega_0}$, Q-a.s.. Conversely, if $Q^p_\omega = P^0_{\omega_0}$, Q-a.s. and $H(Q) < \infty$, then $H_0(Q) = 0$.

Remark 1. (i) If c_0 is strictly positive, then from Lemma 4.5 of ref. 5 we see that either $H(Q) = \infty$ or $H^0(Q) < \infty$, hence $H_0 \equiv H^0$.

(ii) (3) is the variational principle in our case. Combined with (1) it implies that both H_0 and H^0 can be zero only at the stationary measures of the system with $H(Q) < \infty$. By now we cannot say that $Q^p_\omega = P^0_{\omega_0}$ implies $HQ) < \infty$. This is not an obvious fact even if the flip rates are strictly positive, compare Theorem 3 and Proposition 5.1 of ref. 5.

(iii) We think that in some cases, $H_0(Q) < H^0(Q)$ may hold for some Q. Some explanations are given in the remark following the proof of Lemma 2.1, see Section 2.

The following result gives our LD upper bounds.

Theorem 2. For any closed subset F of $M_s(\Omega)$,

$$\limsup_{n\to\infty} \frac{1}{n^{d+1}} \log \sup_{\eta\in E} P^{0,n}_\eta(R_n \in F) \leqslant -\inf_{Q\in F} H_0(Q)$$

The same conclusion is also true with $P^{0,n}_\eta$ replaced by P^0_η.

As we stated previously, to obtain nontrivial lower bounds, some conditions should be imposed on the initial configurations. To give such a condition, we first notice that $\{P^{0,n}_\eta, \eta \in E\}$ is in fact a continuous time Markov Chain with the finite state space E_n, $P^{0,n}_\eta$ depends only on $\eta|_{\Lambda_n}$, so η is regarded as in E_n for this process. Denote by $p_n(t, \eta, A)$ the corresponding transition probability function. We state the following condition:

(H_1) There exist $A_n \subset E_n$ for all large n such that

$$\lim_{n \to \infty} \frac{1}{n^{d+1}} \log \inf\{p_n(1, \eta, \xi): \eta \in A_n, \xi \in E_n\} = 0 \tag{1.4}$$

Next, for $Q \in M_s(\Omega)$ and $n \geqslant 1$, define a probability measure μ_Q^n on E_n by

$$\mu_Q^n(A) = Q(\omega_0|_{A_n} \in A), \qquad A \subset E_n$$

Let $M_s^e(\Omega)$ be the set of ergodic measures in $M_s(\Omega)$. We give the second assumption:

(H_2) For every $Q \in M_s^e(\Omega)$, except for at most one, there exist $A_n \subset E_n (n \geqslant 1)$ satisfying (1.4), such that

$$\liminf_{n \to \infty} \mu_Q^n(A_n) > 0 \tag{1.5}$$

Then we have the following

Theorem 3. Assume (H_1) and (H_2). If $A_n^0 \subset E_n (n \geqslant 1)$ is a sequence satisfying (1.4), then for every open subset G of $M_s(\Omega)$,

$$\liminf_{n \to \infty} \frac{1}{n^{d+1}} \log \inf_{\eta \in A_n^0} P_\eta^{0,n}(R_n \in G) \geqslant -\inf_{Q \in G} H^0(Q)$$

Remark 2. Condition (H_1) means that starting from $\eta \in A_n$, the probabilities that the process on E_n can reach any state in E_n by time 1 are not too small. We believe that it is not hard to be satisfied for most systems without absorbing states. In particular, if

$$\lim_{n \to \infty} \frac{1}{n^{d+1}} \log \inf\{p_n(1, \eta, \xi), \eta, \xi \in E_n\} = 0$$

then both (H_1) and (H_2) are satisfied. Note that the above condition is satisfied by any system with strictly positive flip rates.

Remark 3. For the basic contact process, it is well known that the identically -1 configuration $\overrightarrow{-1}$ is the only trap. Using three mutually independent Poisson processes to construct the system (cf. ref. 8), we can check that both (H_1) and (H_2) are satisfied. Indeed, we can choose $A_n = \{\eta \in E_n, \sum_{i \in A_n} \eta(i) > -n^d\}$ in (H_1) and see that for every $Q \in M_s^e(\Omega)$, except for $\delta_{\vec{1}}$, we have $\mu_Q^n(A_n) \to 1$. Therefore for every $\eta \neq -\vec{1}$, every open $G \subset M_s(\Omega)$,

$$\liminf_{n \to \infty} \frac{1}{n^{d+1}} \log P_\eta^{0,n}(R_n \in G) \geqslant - \inf_{Q \in G} H^0(Q)$$

Remark 4. From our arguments in the following sections we believe that if the flip rates are easy to be zero, we cannot expect to obtain an LDP with H_0 governing the lower bounds, even if the system has no traps.

Remark 5. From Proposition 4.1 of ref. 5 we see that if $H(Q) < \infty$, then

$$\begin{aligned} H^0(Q) &= \limsup_{n \to \infty} \frac{1}{n^2} E^Q \left[\log \left(\frac{dQ_\omega^p}{dP_{\omega_0}} \bigg|_{F_n} \right) - \log \left(\frac{dP_{\omega_0}^{0,n}}{dP_{\omega_0}} \bigg|_{F_n} \right) \right] \\ &= H(Q) - \liminf_{n \to \infty} \frac{1}{n^2} E^Q \log Z_n(\omega^n) \end{aligned}$$

and from Corollary 4.4 of ref. 5 we know that

$$\frac{dP_{\omega_0}^0}{dP_{\omega_0}} \bigg|_{F_n} (\omega) \leqslant Z_n^{\omega_n(\omega)}(\omega)$$

for some function $\omega_n(\omega)$ of ω, for the definitions of $Z_n(\omega^n)$ and $Z_n^{\omega_n(\omega)}(\omega)$, see Section 2. Thus if we replace $Z_n(\omega^n)$ by $Z_n^{\omega_n}(\omega)$, we can similarly define a function $H^{0,\prime}$ which will govern the LD lower bounds for $\{P_\eta^0(R_n \in \cdot),\ n \geqslant 1\}$. Systems with more general boundary conditions can also be discussed. But it seems that the governing functions defined in this way may be different. The effects of the boundary conditions may appear here. We will not give further discussions. In the following we only apply the upper bounds to attractive systems.

Recall that a spin system $\{c(i, \eta),\ i \in \mathbf{Z}^d,\ \eta \in E\}$ is said to be attractive, if for η and ξ in E with $\eta \leqslant \xi$ (i.e., $\eta(i) \leqslant \xi(i)$, $\forall i \in \mathbf{Z}^d$) we have

$$\begin{cases} c(i, \eta) \leqslant c(i, \xi), & \text{if } \eta(i) = \xi(i) = -1, \\ c(i, \eta) \geqslant c(i, \xi), & \text{if } \eta(i) = \xi(i) = 1 \end{cases}$$

It is well known that for a translation invariant and attractive system, there exist a lower and an upper stationary distributions ν_- and ν_+ for the process. The system is ergodic iff $\nu_- = \nu_+$. If we define $\rho_\pm = \nu_\pm(\eta(0) = 1)$, then $\rho_- \leqslant \rho_+$, and the system is ergodic iff $\rho_- = \rho_+$ (cf. ref. 10, Chap. 3).

From the point of view of ergodicity, we need only study the asymptotic behavior of the occupation times of the system on each finite set of sites. So we define

$$T_t = T_t(\omega) = \frac{1}{t} \int_0^t \omega_s(0)\, ds, \qquad t > 0$$

Then we have the following

Theorem 4. Given an attractive system with translation invariant flip rates. Let $\rho_\pm$ be defined as above. Then for any $\delta > 0$, there exists $\gamma_\delta > 0$ such that

$$\sup_{\eta \in E} P_\eta^0(T_t > \rho_+ + \delta \text{ or } T_t < \rho_- - \delta) \leqslant e^{-\gamma_\delta t}, \qquad \forall t > 0$$

In particular, if the system is ergodic, $\rho_- = \rho_+ = \rho$, then for any $\delta > 0$

$$\sup_{\eta \in E} P_\eta^0(|T_t - \rho| > \delta) \leqslant e^{-\gamma_\delta t}, \qquad \forall t > 0$$

Therefore for any $\eta \in E$, $T_t \to \rho$, P_η-a.s. as $t \to \infty$.

Several authors have studied the occupation time large deviations for some special particle systems, see refs. 1, 3, 9 and 10. They mainly considered some special initial distributions, e.g., product measures or invariat measures of the systems, which make it possible to give more explicite computations or estimates, hence the results obtained are critical in some cases. Our Theorem 4 applies to all attractive systems with arbitrary initial distributions and, as can be easily seen, provides new information especially to ergodic systems. For example, for the supercritical and subcritical basic contact processes, some convergence results for the occupation times with certain initial distributions can be deduced from the results of ref. 8 and 11, but if the initial distributions is arbitrary or if the system is critical, little is known by now. Combined with the result of ref. 2, our result means that $T_t \to 0$ exponentially fast for the critical contact process.

On the other hand, even for most of the special systems considered in refs. 1, 3 and 9, the full LDP has not been proved. This may also indicate that it is difficult to obtain such kind of results.

Theorems 1–4 are proved in the next four sections respectively. The proofs of Theorems 1 and 2 involve comparison with a family of systems with strictly positive flip rates, so that the results obtained in refs. 4 and 5 can be applied. This approach can not be used to prove Theorem 3. We will use a technique initiated by Donsker and Varadhan (cf. ref. 12).

Note. To make the notations used in the proofs simple, we work in the case $d = 1$.

2. THE RATE FUNCTIONS

We first define a family of systems with strictly positive flip rates which will be used in both this and the next sections. Let

$$r_0 = \min\{c_0(\eta), \eta \in E, c_0(\eta) \neq 0\}$$

Then $r_0 > 0$. For each $r > 0$, define a system by

$$c^r(i, \eta) = c(i, \eta) \vee r = c_0(\theta_i \eta) \vee r$$

where $a \vee b = \max(a, b)$. A simple but useful fact is that, $\forall r \in (0, r_0)$,

$$c^r(i, \eta) = c(i, \eta), \qquad \text{whenever} \quad c(i, \eta) \neq 0 \tag{2.1}$$

Denote by $\{P^r_\eta, \eta \in E\}$ and $\{P^{r,n}_\eta, \eta \in E\}$ the Markov processes determined by (1.1) and (1.2) respectively, with $c(i, \eta)$ replaced by $c^r(i, \eta)$. Then $\forall r \geqslant 0$, $P^{r,n}_\eta|_{F_n} \ll P_\eta|_{F_n}$ and

$$\left.\frac{dP^{r,n}_\eta}{dP_\eta}\right|_{F_n}(\omega) = Z^r_n(\omega^n)$$

where for $\omega \in \Omega$ and $r \geqslant 0$,

$$Z^r_n(\omega) = \exp\left\{\sum_{i \in \Lambda_n} \int_0^n [1 - c^r(i, \omega_s)]\, ds + \sum_{i \in \Lambda_n} \int_0^n \log c^r(i, \omega_{s-})\, N_i(ds)\right\}$$

and ω^n is the periodic element of ω, as defined is §1 (cf. ref. 4). For ω and ω' in Ω, we also define

$$Z^{r,\omega'}_n(\omega) = \exp\left\{\sum_{i \in \Lambda_n} \int_0^n [1 - c^r(i, (\omega\omega')_s)]\, ds + \sum_{i \in \Lambda_n} \int_0^n \log c^r(i, (\omega\omega')_{s-})\, N_i(ds)\right\}$$

with $\omega\omega' \in \Omega$ satisfying $(\omega\omega')_t(i) = \omega_t(i)$, if $i \in \Lambda_n$; $= \omega'_t(i)$, if $i \in \Lambda^c_n$. We simply write $Z_n = Z^0_n$ and $Z^{\omega'}_n = Z^{0,\omega'}_n$. it is easy to show that

$$Z^r_n(\omega) \leqslant Z_n(\omega) \leqslant Z^r_n(\omega)\, e^{rn^{d+1}}, \qquad \text{whenever} \quad Z_n(\omega) \neq 0 \tag{2.2}$$

The same inequality also holds with Z_n^r and Z_n replaced by $Z_n^{r,\omega'}$ and $Z_n^{\omega'}$ respectively.

Now we start to proove Theorem 1. Let $U_n = \{i \in \Lambda_n : i + U_0 \not\subset \Lambda_n\}$ and define $I_n(Q) = E^Q \log Z_n(\omega)$. Then from Proposition 4.1 of ref. 5, when $H(Q) < \infty$, we can write

$$H^0(Q) = H(Q) - \liminf_{n \to \infty} \frac{1}{n^2} I_n(Q)$$

Thus to prove conclusion (1), it suffices to prove the following

Lemma 2.1. If $H^0(Q) < \infty$, then $\lim_{n \to \infty}(1/n^2)\, I_n(Q)$ exists and

$$-\infty < \lim_{n \to \infty} \frac{1}{n^2} I_n(Q) = E^Q\left[1 - c_0(\omega_0) + \int_0^1 \log c_0(\omega_{s-})\, N_0(ds)\right]$$

Proof. Define for $r \geqslant 0$

$$J_r(Q) = E^Q\left[1 - c_0^r(\omega_0) + \int_0^1 \log c_0^r(\omega_{s-})\, N_0(ds)\right]$$

Since $H^0(Q) < \infty$, $\liminf_{n \to \infty}(1/n^2)\, I_n(Q) > -\infty$. Thus from the definition of $Z_n(\omega^n)$ we know that for all large n,

$$E^Q \sum_{i \in \Lambda_n} \int_0^n \log c(i, \omega_{s-}^n)\, N_i(ds) > -\infty$$

and hence from (2.1) we see that for large n and $r \in (0, r_0)$,

$$\begin{aligned}
&E^Q \sum_{i \in \Lambda_n} \int_0^n \log c(i, \omega_{s-}^n)\, N_i(ds) \\
&\quad = E^Q \sum_{i \in \Lambda_n} \int_0^n \log c_0^r(i, \omega_{s-}^n)\, N_i(ds) \\
&\quad = E^Q\left[\sum_{i \in \Lambda_n - U_n} \int_0^n \log c^r(i, \omega_{s-})\, N_i(ds) + \sum_{i \in U_n} \int_0^n \log c^r(i, \omega_{s-}^n)\, N_i(ds)\right]
\end{aligned}$$

From the stationariness of Q it follows that $\forall r \in (0, r_0)$ and for large n,

$$\begin{aligned}
\frac{1}{n^2} I_n(Q) - J_r(Q) = \frac{1}{n^2} E^Q \Bigg\{ &\sum_{i \in \Lambda_n} \int_0^n [c^r(i, \omega_s) - c(i, \omega_s^n)]\, ds \\
&+ \sum_{i \in U_n} \int_0^n [\log c^r(i, \omega_{s-}^n) - \log c^r(i, \omega_{s-})]\, N_i(ds) \Bigg\}
\end{aligned}$$

Noticing that for $i \in \Lambda_n - U_n$ and $0 \leqslant s < n$,

$$c(i, \omega_s^n) = c(i, \omega_s) \leqslant c^r(i, \omega_s) \leqslant c(i, \omega_s) + r, \qquad \forall r > 0$$

we see that for $r \in (0, r_0)$,

$$\begin{aligned} -\frac{|U_n|}{n}&\left[\|c_0\|_\infty + \|\log c_0^r\|_\infty E^Q \int_0^1 N_0(dt)\right] \\ &\leqslant \frac{1}{n^2} I_n(Q) - J_r(Q) \\ &\leqslant r + \frac{|U_n|}{n}\left[\|c_0\|_\infty + \|\log c_0^r\|_\infty E^Q \int_0^1 N_0(dt)\right] \end{aligned}$$

By Proposition 4.5 of ref. 5 we know that $H(Q) < \infty$ implies $E^Q \int_0^1 N_0(dt) < \infty$, thus

$$-\infty < J_r(Q) \leqslant \liminf_{n\to\infty} \frac{1}{n^2} I_n(Q) \leqslant \limsup_{n\to\infty} \frac{1}{n^2} I_n(Q) \leqslant r + J_r(Q)$$

Using the Dominated Convergence Theorem we have $\lim_{r \searrow 0} J_r(Q) = J(Q)$. Therefore,

$$-\infty < \lim_{n\to\infty} \frac{1}{n^2} I_n(Q) = J(Q)$$

proving the lemma.

Here we try to give some more considerations. Since

$$\frac{1}{n^2} I_n(Q) = E^Q[X_n + X_n^{(1)} + Y_n^{(1)}]$$

and $J(Q) = 1/n^2\, E^Q[X_n + X_n^{(2)} + Y_n^{(2)}]$ where $X_n = \sum_{i\in\Lambda_n - U_n} \int_0^n 1 - c(i, \omega_s)\, ds + \sum_{i\in\Lambda_n - U_n} \int_0^n \log c(i, \omega_{s-})\, N_i(ds)$, $X_n^{(1)} = \sum_{i\in U_n} \int_0^n [1 - c(i, \omega_s^n)]\, ds$, $Y_n^{(1)} = \sum_{i\in U_n} \int_0^n \log c(i, \omega_{s-}^n)\, N_i(ds)$, $X_n^{(2)}$ and $Y_n^{(2)}$ are defined similarly, with ω^n replaced by ω. Noting

$$\frac{1}{n^2} |X_n^{(1)} - X_n^{(2)}| \leqslant \frac{|U_n|}{n} \|c_0\|_\infty \to 0$$

we see that if $H(Q) < \infty$ and

$$E^Q Y_n^{(1)} = E^Q Y_n^{(2)}$$

for all large n, then $H_0(Q) = H^0(Q)$. Thus if for some Q with $H(Q) < \infty$

$$E^Q Y_{n_k}^{(1)} = -\infty < E^Q Y_{n_k}^{(2)} \tag{2.3}$$

for some subsequence n_k, then $H_0(Q) < H^0(Q)$. Since we know there are some cases for which (2.3) may hold for some Q, we think that $H_0(Q) < H^0(Q)$ may hold for some Q.

To prove conclusion (2), we notice that for $r > 0$ and $Q \in M_s(\Omega)$,

$$H_r(Q) \equiv H(Q) - J_r(Q) \leqslant H_0(Q) + r$$

From ref. 5 we know that both H and H_r are l.s.c. with compact level sets, so if $Q_n \Rightarrow Q$ $(n \to \infty)$, we have

$$\liminf_{n \to \infty} H_0(Q_n) \geqslant H_r(Q) - r$$

Since $\lim_{r \searrow 0} H_r(Q) = H_0(Q)$, we see that H_0 is l.s.c.. Furthermore, since $\{H_0(Q) \leqslant a\} \subset \{H_r(Q) \leqslant a + r\}$, H_0 has compact level sets.

Finally we prove conclusion (3). First define for $n \geqslant 1$

$$G_n = \sigma(\omega_t(i), 0 \leqslant t \leqslant 1, i \in \Lambda_n), \qquad \text{and} \qquad G^n = \sigma(\omega_t(i), 0 \leqslant t \leqslant 1, i \in \Lambda_n^c)$$

For a probability measure R on $D([0, 1], E)$, denote by $R_{G^n}|_{G_n}$ the r.c.p.d. of $R|_{G_n}$ given G^n. To prove that $H_0(Q) = 0$ implies $Q_\omega^P = P_{\omega_0}^0$ Q-a.s., it is sufficient to show

$$Q_{\omega, G^n}^P|_{G_n} = P_{\omega_0, G^n}^0|_{G_n} \qquad \text{and} \qquad Q_{\omega, G^n}^P|_{G_n} \ll P_{\omega_0}|_{G_n} \tag{2.4}$$

see the proof of Theorem 6 in ref. 5. The proof of (2.4) will be completed with several lemmas.

Lemma 2.2. Let $Q \in M_s(\Omega)$ be such that $H(Q) < \infty$. There exists a constant $\rho > 0$, such that

$$E^Q \log\left(\frac{dQ_\omega^P}{dP_{\omega_0}}\bigg|_{G_n}\right) \leqslant n\rho, \qquad \forall n \geqslant 1$$

Proof. By Propopisition 5.2 of ref. 5, $H(Q) < \infty$ implies

$$H(Q) = \lim_{n \to \infty} \frac{1}{n} E^Q \log\left(\frac{dQ^P_\omega}{dP_{\omega_0}} \bigg|_{G_n} \right)$$

The conclusion follows.

Lemma 2.3. If $H_0(Q) = 0$, then for each $n \geqslant 1$, we can choose a sequence of positive numbers $\{r_k\}$ decreasing to 0, such that

$$E^Q \log\left(\frac{dQ^P_{\omega, G^n}}{dP_{\omega_0}} \bigg|_{G_n} \right) = \lim_{k \to \infty} E^Q \log\left(\frac{dP^{r_k}_{\omega_0, G^n}}{dP_{\omega_0}} \bigg|_{G_n} \right)$$

Proof. We first show that we can choose $r_k \searrow 0$ and a sequence of finite subsets W_k of $\mathbf{Z}$ increasing to Λ_n^c such that

$$\lim_{k \to \infty} E^Q \log\left(\frac{dQ^P_{\omega, G_{W_k}}}{dP^{r_k}_{\omega_0, G_{W_k}}} \bigg|_{G_n} \right) = 0 \tag{2.5}$$

where $G_{W_k} = \sigma(\omega_t(i),\ 0 \leqslant t \leqslant 1,\ i \in W_k)$. Since $H_0(Q) = 0$, it follows from (2.1) that $H_r(Q) \leqslant r$ for $r > 0$. For $k \geqslant 1$. Let

$$\partial_k \Lambda_n = \{ j \in \Lambda_n^c : d(j, \Lambda_n) = \min_{i \in \Lambda_n} |j - i| \leqslant k \}$$

For every $m \geqslant 1$, denote $N_m = (n + 2k)\, m$. Then repeating the proof of Theorem 3.31 of ref. 6 we see that for $r_k > 0$,

$$\frac{n + 2k}{N_m} E^Q \log\left(\frac{dQ^P_\omega}{dP^{r_k}_{\omega_0}} \bigg|_{G_{N_m}} \right) \geqslant \frac{1}{m} \sum_{i=1}^{m} E^Q \log\left(\frac{dQ^P_{\omega, G_{V_i}}}{dP^{r_k}_{\omega_0, G_{V_i}}} \bigg|_{G_n} \right) \tag{2.6}$$

where $\partial_k \Lambda_n \subset V_i \subset \Lambda_n^c$ and $G_{V_i} = \sigma(\omega_t(j)\colon 0 \leqslant t \leqslant 1,\ j \in V_i)$. By Proposition 5.2 of ref. 5,

$$\lim_{m \to \infty} \frac{1}{N_m} E^Q \log\left(\frac{dQ^P_\omega}{dP^{r_k}_{w_0}} \bigg|_{G_{N_m}} \right) = H_{r_k}(Q) \leqslant r_k$$

it then follows from (2.6) that we can choose $r_k \searrow 0$ and finite sets $W_k \uparrow \Lambda_n^c$ such that (2.5) holds. Furthermore, from the proof of Theorem 5 of ref. 5 it can be seen that for all sufficiently large k,

$$\frac{dP^{r_k}_{\omega_0, G_{W_k}}}{dP_{\omega_0}} \bigg|_{G_n} = \frac{dP^{r_k}_{\omega_0, G^n}}{dP_{\omega_0}} \bigg|_{G_n}$$

From this and (2.5) we get

$$
\begin{aligned}
E^Q \log\left(\frac{dQ^P_{\omega, G^n}}{dP_{\omega_0}}\bigg|_{G_n}\right)
&= \lim_{k\to\infty}\left[E^Q \log\left(\frac{dQ^P_{\omega, G_{W_k}}}{dP^{r_k}_{\omega_0, G_{W_k}}}\bigg|_{G_n}\right) + E^Q \log\left(\frac{dP^{r_k}_{\omega_0, G_{W_k}}}{dP_{\omega_0}}\bigg|_{G_n}\right)\right] \\
&= \lim_{k\to\infty} E^Q \log\left(\frac{dP^{r_k}{}_{\omega_0, G^n}}{dP_{\omega_0}}\bigg|_{G_n}\right)
\end{aligned}
$$

proving the lemma.

The next goal is to show that for $n \geqslant 1$,

$$
\lim_{k\to\infty} E^Q \log\left(\frac{dP^{r_k}_{\omega_0, G^n}}{dP_{\omega_0}}\bigg|_{G_n}\right) = E^Q \log\left(\frac{dP^0_{\omega_0, G^n}}{dP_{\omega_0}}\bigg|_{G_n}\right) \tag{2.7}
$$

To this end we need to estimate $(dP^0_{\omega_0, G^n}/dP_{\omega_0})|_{G_n}$. For $\omega \in \Omega$, denote by $\omega_{(n)}$ and $\omega^{(n)}$ the projections of ω on $D(\mathbf{R}, E_n)$ and $D(\mathbf{R}, E^c_n)$ respectively. Then define for $n, k \geqslant 1$,

$$
\begin{aligned}
\psi_{n,k}(\omega) &= \psi_{n,k}(\omega_{(n)}\omega^{(n)}) \\
&= \sum_{i\in U_n} \int_0^1 [1 - c^{r_k}(i, \omega_t)]\, dt + \sum_{i\in U_n} \int_0^1 \log c^{r_k}(i, \omega_{t^-})\, N_i(dt)
\end{aligned}
$$

$$
Z_{n,k}(\omega) = Z_{n,k}(\omega^{(n)}) = \int \exp\{\psi_{n,k}(\omega)\}\, P_{\omega_0}(d\omega_{(n)})
$$

$\psi_{n,0}$ and $Z_{n,0}$ are defined with c^{r_k} replaced by c. Then

$$
\frac{dP^{r_k}_{\omega_0, G^n}}{dP_{\omega_0}}\bigg|_{G_n}(\omega) = Z^{-1}_{n,k}(\omega)\exp\{\psi_{n,k}(\omega)\}
$$

and

$$
\frac{dP^0_{\omega_0, G^n}}{dP_{\omega_0}}\bigg|_{G_n}(\omega) = Z^{-1}_{n,0}\exp\{\psi_{n,0}(\omega)\}\, I_{\{Z_{n,0}\neq 0\}}
$$

see the proof of Theorem 4 in ref. 5, and notice that $Z_{n,0}(\omega)=0$ implies $\psi_{n,0}(\omega)=0$ for P_{ω_0} almost all $\omega_{(n)}$. Thus

$$E^Q \log\left(\frac{dP^{r_k}_{\omega_0, G^n}}{dP_{\omega_0}}\bigg|_{G_n}\right) = E^Q \psi_{n,k} - E^Q \log Z_{n,k}$$

and we have the following

Lemma 2.4. If $H_0(Q)=0$, then $Q(\omega\colon Z_{n,0}(\omega)=0)=0$. Thus

$$E^Q \log\left(\frac{dP^{0}_{\omega_0, G^n}}{dP_{\omega_0}}\bigg|_{G_n}\right) = E^Q \psi_{n,0} - E^Q \log Z_{n,0}$$

and $E^Q \log Z_{n,0} = \lim_{k\to\infty} E^Q \log Z_{n,k} > -\infty$. In particular, (2.7) holds.

Proof. Notice that if $c(i, \omega_{t^-})=0$ for some $i \in U_n$ and $0 \leqslant t \leqslant 1$, then $c^r(i, \omega_{t^-}) = r$. Hence if $Z_{n,0}(\omega)=0$, then for large k,

$$Z_{n,k}(\omega) \leqslant r_k \alpha_n$$

where

$$0 < \alpha_n = e^{\|1-c_0\|_\infty |U_n|} \prod_{i\in U_n} \int \exp\left\{\|\log c_0^1\|_\infty \int_0^1 N_i(dt)\right\} dP_{\omega_0(i)} < \infty$$

If $Z_{n,0} \neq 0$, then $Z_{n,k} \leqslant \alpha_n$ for large k. These mean that if $Q(Z_{n,0}(\omega)=0)>0$, then

$$E^Q \log Z_{n,k} \leqslant (\log r_k)\, Q(Z_{n,0}=0) + \log \alpha_n \to -\infty, \qquad (k\to\infty) \tag{2.8}$$

Since $H_0(Q)<\infty$, $E^Q \log \psi_{n,0}$ is finite, thus

$$E^Q \log \psi_{n,k} \to E^Q \log \psi_{n,0}, \qquad (k\to\infty) \tag{2.9}$$

Combining this with (2.8) we see that

$$E^Q \log\left(\frac{dP^{r_k}_{\omega_0, G^n}}{dP_{\omega_0}}\bigg|_{G_n}\right) \to +\infty, \qquad (k\to\infty) \tag{2.10}$$

On the other hand, from Lemma 2.2 and the fact that $E^Q \log((dQ^P_\omega / dP^{r_k}_{\omega_0, G_n})|_{G_n}) \geqslant 0$, it follows that for $n \geqslant 1$

$$\sup_k E^Q \log\left(\frac{dP^{r_k}_{\omega_0, G_n}}{dP_{\omega_0}}\bigg|_{G_n}\right)$$
$$= \sup_k \left[E^Q \log\left(\frac{dQ^P_\omega}{dP_{\omega_0}}\bigg|_{G_n}\right) - E^Q \log\left(\frac{dQ^P_\omega}{dP^{r_k}_{\omega_0, G^n}}\bigg|_{G_n}\right)\right] \leqslant n\rho$$

contradicting (2.10). So $Q(\omega: Z_{n,0}(\omega) = 0) = 0$. Therefore, by the definition of $Z_{n,0}$ we have

$$P_{\omega_0}(\omega_{(n)}: \psi_{n,0}(\omega_{(n)}\omega^{(n)}) > -\infty) = 1, \qquad Q\text{-a.s.}$$

Thus Q-a.s. for large k with $r_k \in (0, r_0)$, we have

$$Z_{n,k}(\omega)\, e^{r_k |U_n|} \geqslant Z_{n,0}(\omega)$$
$$= \int \exp\left\{ \sum_{i \in U_n} \int_0^1 [1 - c(i, \omega_t)\, dt \right.$$
$$\left. + \sum_{i \in U_n} \int \log c^{r_k}(i, \omega_{t-})\, N_i(dt) \right\} P_{\omega_0}(d\omega_{(n)})$$
$$\geqslant Z_{n,k}$$

This means that $E^Q \log Z_{n,0}$ is finite and that we can use the Dominated convergence theorem to conclude that

$$E^Q \log Z_{n,0} = \lim_{k \to \infty} E^Q \log Z_{n,k}$$

Now (2.7) follows from this and (2.9), completing the proof.

From Lemma 2.3 and (2.7) it follows that

$$0 \leqslant E^Q \log\left(\frac{dQ^P_{\omega, G^n}}{dP_{\omega_0}}\bigg|_{G_n}\right) = E^Q \log\left(\frac{dP^0_{\omega, G^n}}{dP_{\omega_0}}\bigg|_{G_n}\right) < \infty$$

This implies $E^Q \log(dQ^P_{\omega, G_n}/dP^0_{\omega_0, G^n})|_{G_n} = 0$ and hence (2.4) holds. Then repeating the proof of Theorem 6 of ref. 5 we obtain $Q^P_\omega = P^0_{\omega_0}$ Q-a.s..

Finally we prove that if $Q_\omega^P = P_{\omega_0}^0$ Q-a.s. and $H(Q) < \infty$, then $H_0(Q) = 0$. By Lemma 4.1 of ref. 5

$$H(Q) = \lim_{n \to \infty} \frac{1}{n^2} E^Q \log \left(\frac{dP_{\omega_0}^0}{dP_{\omega_0}} \bigg|_{F_n} \right)$$

Thus by the definition of H_0 we will obtain that $H_0(Q) = 0$ once we show that

$$\lim_{n \to \infty} \frac{1}{n^2} E^Q \log \left(\frac{dP_{\omega_0}^0}{dP_{\omega_0}} \bigg|_{F_n} \right) \leqslant J(Q) \tag{2.11}$$

To prove (2.11), we use the stationariness of Q to obtain that for $r \in (0, r_0)$,

$$\begin{aligned}
\alpha_{r,n} &\equiv J_r(Q) - \frac{1}{n^2} E^Q \log \left(\frac{dP_{\omega_0}^0}{dP_{\omega_0}} \bigg|_{F_n} \right) \\
&= \frac{1}{n^2} E^Q \left\{ E^{Q_\omega^P} \left[\sum_{i \in \Lambda_n} \int_0^n (1 - c^r(i, \omega'_t))\, dt \right. \right. \\
&\quad \left. \left. + \sum_{i \in \Lambda_n} \int_0^n \log c^r(i, \omega'_{t-})\, N_i(dt) - \log \left(\frac{dP_{\omega_0}^0}{dP_{\omega_0}} \bigg|_{F_n} \right) \right] \right\} \\
&= \frac{1}{n^2} E^Q E^{P_{\omega_0}} \left\{ \left[\sum_{i \in \Lambda_n} \int_0^n (1 - c^r(i, \omega'_t))\, dt + \sum_{i \in \Lambda_n} \int_0^n \log c^r(i, \omega'_{t-})\, N_i(dt) \right. \right. \\
&\quad \left. \left. - \log \left(\frac{dP_{\omega_0}^0}{dP_{\omega_0}} \bigg|_{F_n} \right) \right] \frac{dP_{\omega_0}^0}{dP_{\omega_0}} \bigg|_{F_n} ; \frac{dP_{\omega_0}^0}{dP_{\omega_0}} \bigg|_{F_n} \neq 0 \right\}
\end{aligned} \tag{2.12}$$

From the proof of Corollary 4.4 of ref. 5 and (2.2) we can see that for $\omega' \in \{(dP_{\omega_0}^0/dP_{\omega_0})|_{F_n}(\omega') \neq 0\}$, there exists $\omega_n = \omega_n(\omega')$ such that

$$\frac{dP_{\omega_0}^0}{dP_{\omega_0}} \bigg|_{F_n} (\omega') \leqslant Z_n^{r, \omega_n}(\omega')\, e^{n^2 r}, \qquad \forall r \in (0, r_0)$$

From this, the definition of Z_n^{r, ω_n} and (2.12) it follows that

$$\alpha_{r,n} \geqslant \frac{1}{n^2} E^Q\, E^{P_{\omega_0}} (\sup_{\omega''} \log Z_n^{r, \omega''} - \inf_{\omega'} \log Z_n^{r, \omega'}) - r$$

By this, the fact that $E^{Q}\int_0^1 N_0(dt)<\infty$ and the proof of Lemma 4.7 of ref. 5 we obtain

$$\lim_{n\to\infty} \alpha_{r,n} \geqslant -r$$

i.e., $J_r(Q)\geqslant \lim_{n\to\infty}(1/n^2)\,E^{Q}\log((dP^0_\omega/dP_{\omega_0})|_{F_n})-r$. Letting $r\searrow 0$ to get (2.11). Theorem 1 is proved.

3. THE UPPER BOUNDS

In this section, we prove Theorem 2. First we observe that $\{\sup_\eta P^{0,n}_\eta(R_n\in\cdot), n\geqslant 1\}$ are exponentially tight, i.e., for any $K>0$, there exists a compact subset C_k of $M_s(\Omega)$, such that

$$\limsup_{n\to\infty}\frac{1}{n^2}\log\sup_\eta P^{0,n}_\eta(R_n\in C^c_K)\leqslant -K$$

since it is true for $\sup_\eta P^{r,n}_\eta(R_n\in\cdot)$ with $r>0$ (cf. refs. 4 and 5) and we have (2.2). Thus to obtain the upper bounds, we need only show that for each $Q_0\in M_s(\Omega)$ and for any $\delta>0$, there is a neighborhood V_{Q_0} of Q_0 such that

$$\limsup_{n\to\infty}\frac{1}{n^2}\log\sup_\eta P^{0,n}_\eta(R_n\in V_{Q_0})$$
$$\leqslant\begin{cases}-H_0(Q_0)+\delta & \text{if } H_0(Q_0)<\infty\\ -1/\delta & \text{if } H_0(Q_0)=\infty\end{cases}\tag{3.1}$$

We only give the proof for the first case, it is similar for the second case. Since $H_0(Q)<\infty$ implies that $E^{Q_0}\int_0^1\log c_0(\omega_{t^-})\,N_0(dt)$ is finite, for $\delta>0$ we can choose a small $r_1\in(0,\delta/3)$ such that

$$E^{Q_0}\int_0^1\log c_0^{r_1}(\omega_{t^-})\,N_0(dt)\leqslant E^{Q_0}\int_0^1\log c_0(\omega_{t^-})\,N_0(dt)+\frac{\delta}{3}\tag{3.2}$$

By the lower semicontinuity of H_{r_1} (ref. 5), we can choose a neighborhood V_{Q_0} of Q_0 such that

$$H_{r_1}(Q)\geqslant H_{r_1}(Q_0)-\frac{\delta}{3},\qquad \forall Q\in\bar{V}_{Q_0}\tag{3.3}$$

Now from Theorem 7.11 of ref. 4, (2.2) and (3.3) we obtain

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n^2} \log \sup_{\eta} P_{\eta}^{0,n}(R_n \in V_{Q_0}) &\leqslant \limsup_{n\to\infty} \frac{1}{n^2} \log \sup_{\eta} P_{\eta}^{r_1,n}(R_n \in V_{Q_0}) + r_1 \\
&\leqslant -\inf_{Q\in \bar{V}_{Q_0}} H_{r_1}(Q) + \frac{\delta}{3} \\
&\leqslant -H_{r_1}(Q_0) + \frac{2\delta}{3}
\end{aligned}$$

By the definition of H_{r_1} (see (2.1) and (3.2)) we get

$$H_{r_1}(Q_0) \geqslant H_0(Q_0) - \frac{\delta}{3}$$

Combining the above two inequalities we obtain (3.1).

The proof of the upper bounds for $\{P_{\eta}^0(R_n \in \cdot), n \geqslant 1\}$ is similar, since we know that $P_{\eta}^0|_{F_n} \ll P_{\eta}|_{F_n}$ and that there is a function $\omega_n = \omega_n(\omega)$ on Ω such that

$$\left.\frac{dP_{\eta}^0}{dP_{\eta}}\right|_{F_n}(\omega) \leqslant Z_n^{\omega_n(\omega)}(\omega)$$

see the proof of Corollary 4.4 of ref. 5.

4. THE LOWER BOUNDS

Now we start to prove Theorem 3.

Lemma 4.1. Suppose $Q \in M_s^e(\Omega)$ and $H^0(Q) < \infty$. Then

$$\lim_{n\to\infty} \frac{1}{n^2} \log \left(\left.\frac{dQ_{\omega}^P}{dP_{\omega_0}^{0,n}}\right|_{F_n} \right) = H^0(Q) = H_0(Q) \qquad \text{in} \quad L^1(Q)$$

Proof. Since $H^0(Q) < \infty$ implies $H(Q) < \infty$, it follows from the proofs of Lemma 6.4 of ref. 4 and Proposition 4.1 of ref. 5 that

$$\lim_{n\to\infty} \frac{1}{n^2} \log \left(\left.\frac{dQ_{\omega}^P}{dP_{\omega_0}}\right|_{F_n} \right) = H(Q) \qquad \text{in} \quad L^1(Q) \tag{4.1}$$

Also from $H^0(Q) < \infty$ we have

$$\lim_{n\to\infty} \frac{1}{n^2} E^Q \log\left(\frac{dP^{0,n}_{\omega_0}}{dP_{\omega_0}}\bigg|_{F_n}\right) = J(Q) > -\infty$$

Thus Q-a.s. $\log(dP^{0,n}_{\omega_0}/dP_{\omega_0})|_{F_n}$ is finite for large n and by (2.1) we know that $\forall r \in (0, r_0)$,

$$\frac{1}{n^2}\log\left(\frac{dP^{0,n}_{\omega_0}}{dP_{\omega_0}}\bigg|_{F_n}\right) = \frac{1}{n^2}\log Z_n(\omega^n) = \frac{1}{n^2}(X^{(1)}_n + X^{(2)}_n), \qquad Q\text{-a.s.}$$

where

$$X^{(1)}_n = \sum_{i\in A_n}\int_0^n [1 - c(i, \omega_s)]\,ds + \sum_{i\in A_n}\int_0^n \log c(i, \omega_{s-})\,N_i(ds)$$

$$X^{(2)}_n = \sum_{i\in U_n}\int_0^n [c(i, \omega_s) - c(i, \omega^n_s)]\,ds$$

$$+ \sum_{i\in U_n}\int_0^n [\log c^r(i, \omega^n_{s-}) - \log c^r(i, \omega_{s-})]\,N_i(ds)$$

From the ergodicity of Q and the fact that $J(Q) = (1/n^2)\,E^Q X^{(1)}_n$ is finite we see that $\lim_{n\to\infty}(1/n^2)\,X^{(1)}_n = J(Q)$ in $L^1(Q)$. Moreover, note that

$$\frac{1}{n^2}E^Q\,|X^{(2)}_n| \leqslant \frac{|U_n|}{n}\left[\|c_0\|_\infty + \|\log c^r_0\|_\infty\,E^Q\int_0^1 N_0(dt)\right] \to 0, \qquad (n\to\infty)$$

These together with (4.1) complete the proof.

Recall the definition of μ^n_Q for $Q \in M_s(\Omega)$ and $n \geqslant 1$. We have the following

Lemma 4.2. Let $Q \in M^e_s(\Omega)$ be such that $H^0(Q) < \infty$, $A_n \subset E_n (n \geqslant 1)$ satisfy $\liminf_{n\to\infty}\mu^n_Q(A_n) > 0$. Then for any sequence $A^0_n \subset E_n (n \geqslant 1)$ satisfying (1.5) and any open $G \ni Q$,

$$\liminf_{n\to\infty}\frac{1}{n^2}\log\inf_{\eta\in A^0_n} P^{0,n}_\eta(R_n \in G, \omega_n|_{A_n} \in A_n) \geqslant -H_0(Q) = -H^0(Q)$$

In particular,

$$\liminf_{n\to\infty}\frac{1}{n^2}\log\inf_{\eta\in A^0_n} P^{0,n}_\eta(R_n \in G) \geqslant -H^0(Q) \tag{4.2}$$

Proof. Notice that for any $\varepsilon > 0$, $k \geqslant 1$ and open $G_1 \ni Q$, when $n > k$,

$$\alpha_n^k \equiv Q\left(R_n \in G_1, \frac{1}{n^2} \log\left(\frac{dQ_\omega^P}{dP_{\omega_0}^{0,n}}\bigg|_{F_n}\right) \leqslant H^0(Q) + \varepsilon, \omega_{n-k}|_{A_n} \in A_n\right)$$

$$\geqslant \mu_Q^n(A_n) - Q\left(\left\{\frac{1}{n^2} \log\left(\frac{dQ_\omega^P}{dP_{\omega_0}^{0,n}}\bigg|_{F_n}\right) \leqslant H^0(Q) + \varepsilon, R_n \in G_1\right\}^c\right)$$

From the ergodicity of Q and Lemma 4.1 we see that $\liminf_{n \to \infty} \alpha_n^k > 0$. Now for $\omega \in \Omega$,

$$P_{\omega_0}^{0,n}(R_n \in G_1, \omega_{n-k}|_{A_n} \in A_n)$$

$$= \int_{R_n \in G_1, \omega_{n-k}|_{A_n} \in A_n} \frac{dP_{\omega_0}^{0,n}}{dQ_\omega^P}\bigg|_{F_n} dQ_\omega^P$$

$$\geqslant \exp\{-n^2[H^0(Q) + \varepsilon]\}$$

$$\times Q_\omega^P\left(R_n \in G_1, \frac{1}{n^2} \log\left(\frac{dQ_\omega^P}{dP_{\omega_0}^{0,n}}\bigg|_{F_n}\right) \leqslant H^0(Q) + \varepsilon, \omega_{n-k}|_{A_n} \in A_n\right)$$

Therefore,

$$\int_{E_n} P_\xi^{0,n}(R_n \in G_1, \omega_{n-k}|_{A_n} \in A_n)\, \mu_Q^n(d\xi)$$

$$= \int P_{\omega_0}^{0,n}(R_n \in G_1, \omega_{n-k}|_{A_n} \in A_n)\, Q(d\omega)$$

$$\geqslant \exp\{-n^2[H^0(Q) + \varepsilon]\}\, \alpha_n^k$$

Thus

$$\liminf_{n \to \infty} \frac{1}{n^2} \log \int P_\xi^{0,n}(R_n \in G_1, \omega_{n-k}|_{A_n} \in A_n)\, \mu_Q^n(d\xi) \geqslant -H^0(Q) \qquad (4.3)$$

Now for open $G \ni Q$, pick a neighborhood G_1 of Q such that $G_1 \subset G$ and choose $m \in \mathbf{Z}_+$ and $\delta > 0$ such that if $Q' \in G_1$ and $\|Q' - Q''\|_{F_m} < \delta$ then $Q'' \in G$. Since

$$\|R_{n,\omega} - R_{n,\theta_{1,0}\omega}\|_{F_m} \to 0 \qquad (n \to \infty)$$

uniformly in ω, from the markov property we have that for large n,

$$\inf_{\eta \in A_n^0} P_\eta^{0,n}(R_n \in G, \omega_n|_{A_n} \in A_n)$$

$$\geqslant \inf_{\eta \in A_n^0} P_\eta^{0,n}(R_{n,\theta_{1,0}\omega} \in G_1, (\theta_{1,0}\omega)_{n-1}|_{A_n} \in A_n)$$

$$= \inf_{\eta \in A_n^0} \int_{E_n} P_\xi^{0,n}(R_{n,\omega} \in G_1, \omega_{n-1}|_{A_n} \in A_n)\, p_n(1, \eta, d\xi)$$

$$\geqslant [\inf_{\eta \in A_n^0, \xi \in E_n} p_n(1, \eta, \xi)]$$

$$\times \int_{E_n} P_\xi^{0,n}(R_n \in G_1, \omega_{n-1}|_{A_n} \in A_n)\, \mu_Q^n(d\xi)$$

Combining this with (1.5) and (4.3) we complete the proof.

It remains to extend (4.2) to general $Q \in M_s(\Omega)$ with $H^0(Q) < \infty$. Notice that for such a Q, $H^0(Q) = H_0(Q)$ and that H_0 is both semi-continuous and affine (since so is H). As done in ref. 4, we may assume $Q = \sum_{i=1}^m \alpha_i Q_i$ with $Q_i \in M_s^e(\Omega)$, $\alpha_i > 0$ and $\sum_{i=1}^m \alpha_i = 1$. To prove (4.2) for this Q, we first remark that for $0 < \alpha < 1$, if we define

$$R_{\alpha n, n}(\omega) = \frac{1}{\alpha n^2} \sum_{i \in A_n} \int_0^{\alpha n} \delta_{\theta_{t,i}\omega^n}\, dt$$

then under the assumptions of Lemma 4.2 and repeating the proof of it we have

$$\liminf_{n \to \infty} \frac{1}{n^2} \log \inf_{\eta \in A_n^0} P_\eta^{0,n}(R_{\alpha n, n} \in G, \omega_{\alpha n}|_{A_n} \in A_n) \geqslant -\alpha H^0(Q)$$

Now we finish the proof of Theorem 3 with the following

Lemma 4.3. Let $Q = \sum_{i=1}^m \alpha_i Q_i$ with $Q_i \in M_s^e(\Omega)$, $\alpha_i > 0$ and $\sum_{i=1}^m \alpha_i = 1$. If $H^0(Q) < \infty$, then for any sequence $A_n^0 \subset E_n (n \geqslant 1)$ satisfying (1.4) and any open set $G \ni Q$,

$$\liminf_{n \to \infty} \frac{1}{n^2} \log \inf_{\eta \in A_n^0} P_\eta^{0,n}(R_n \in G) \geqslant -H^0(Q)$$

Proof. From assumption (H_2), we may assume that for $1 \leqslant i \leqslant m-1$, there exist $A_n^i \subset E_n (n \geqslant 1)$ such that Q_i and A_n^i satisfy (1.5). Pick a neighborhood G_i of Q_i for each i, and choose $\delta > 0$ and $N \in \mathbf{Z}_+$, such that if $Q'_i \in G_i$ for $1 \leqslant i \leqslant m$ and $\|Q' - \sum_{i=1}^m \alpha_i Q'_i\|_{F_N} < \delta$, then $Q' \in G$. Denote by $\alpha_0 = 0$, $\lambda_i = \sum_{j=0}^i \alpha_j$ and define

$$R_n^i(\omega) = \frac{1}{\alpha_i n^2} \sum_{j \in A_n} \int_{\lambda_{i-1} n}^{\lambda_i n} \delta_{\theta_{t,\, j}\omega^n}\, dt, \qquad 1 \leqslant i \leqslant m$$

Then it is not difficult to check that

$$\|R_n^i(\omega) - R_{\alpha_i n,\, n}(\theta_{\lambda_{i-1} n}\omega)\|_{F_N} \to 0 \qquad (n \to \infty)$$

uniformly in ω and i. From this and the Markov property we have

$$\begin{aligned}
\inf_{\eta \in A_n^0} P_\eta^{0,n}(R_n \in G) &\geqslant \inf_{\eta \in A_n^0} P_\eta^{0,n}(R_{\alpha_i n,\, n}(\theta_{\lambda_{i-1} n,\, 0}\omega) \in G_i,\ \omega_{\lambda_i n}|_{A_n} \in A_n^i, \\
&\qquad 1 \leqslant i \leqslant m-1,\ R_{\alpha_m n,\, n}(\theta_{\lambda_{m-1} n,\, 0}\omega) \in G_m) \\
&\geqslant \left[\prod_{i=1}^{m-1} \inf_{\eta \in A_n^{i-1}} P_\eta^{0,n}(R_{\alpha_i n,\, n} \in G_i,\ \omega_{\alpha_i n}|_{A_n} \in A_n^i)\right] \\
&\qquad \times \inf_{\eta \in A_n^{m-1}} P_\eta^{0,n}(R_{\alpha_m n,\, n} \in G_m)
\end{aligned}$$

Since $H^0(Q_i) = H_0(Q_i) < \infty$, from the above inequality and the remark following Lemma 4.2 we obtain

$$\liminf_{n \to \infty,\, \eta \in A_n} \frac{1}{n^2} \log \inf_{\eta \in A_n^0} P_\eta^{0,n}(R_n \in G) \geqslant -\sum_{i=1}^m \alpha_i H^0(Q_i) = -H^0(Q)$$

Theorem 3 is proved by now.

5. OCCUPATION TIMES FOR ATTRACTIVE SYSTEM

Finally we prove Theorem 4. For $\omega \in \Omega$ and $n \geqslant 1$, let

$$M_n = M_n(\omega) = \frac{1}{n^2} \sum_{i \in A_n} \int_0^n \omega_t(i)\, dt$$

and $J(x) = \inf\{H_0(Q): \int \omega_0(0)\, Q(d\omega) = x\}$. Then define

$$m_- = \inf\{x: J(x) = 0\} \qquad \text{and} \qquad m_+ = \sup\{x: J(x) = 0\}$$

By the contraction principle (cf. ref. 12) and Theorem 2 we know that $\{P^0_\eta(M_n \in \cdot), n \geqslant 1\}$ has the uniform LDP upper bounds with rate function J. Therefore for any $\delta > 0$, there exists $\gamma_\delta > 0$ such that

$$\sup_\eta P^0_\eta(M_n > m_+ + \delta \text{ or } M_n < m_- - \delta) \leqslant e^{-\gamma_\delta n^2}, \qquad \forall n \geqslant 1 \tag{5.1}$$

Since $\{c(i, \eta), i \in \mathbf{Z}, \eta \in E\}$ is attractive, from Corollary 2.21 of Chap. 2 of ref. 10 we know that the joint distribution of $\{(1/n) \int_0^n \omega_s(i)\, ds, i \in \Lambda_n\}$ under P^0_η has positive correlations. Therefore if we use $\vec{1}$ and $-\vec{1}$ to denote the identically 1 and -1 configurations respectively, then

$$\begin{aligned} P^0_{\vec{1}}(M_n > m_+ + \delta) &\geqslant P^0_{\vec{1}}\left(\frac{1}{n} \int_0^n \omega_s(i)\, ds > m_+ + \delta, \forall i \in \Lambda_n\right) \\ &\geqslant \left[P^0_{\vec{1}}\left(\frac{1}{n} \int_0^n \omega_s(0)\, ds > m_+ + \delta\right)\right]^n \\ &= [P^0_{\vec{1}}(T_n > m_+ + \delta)]^n \end{aligned}$$

where we have used the translation invariance of the system. Hence by the attractiveness and (5.1) we obtain

$$\sup_\eta P^0_\eta(T_n > m_+ + \delta) \leqslant P^0_{\vec{1}}(T_n > m_+ + \delta) \leqslant e^{-\gamma_\delta n}, \qquad \forall n \geqslant 1 \tag{5.2}$$

Using $P^0_{-\vec{1}}$ we can similarly show that

$$\sup_\eta P^0_\eta(T_n < m_- - \delta) \leqslant e^{-\gamma_\delta n}, \qquad \forall n \geqslant 1 \tag{5.3}$$

The extension to general time parameter t is easy. Thus to finish the proof of Theorem 4. We need only show that $m_- = \rho_-$ and $m_+ = \rho_+$. From Theorem 1 and the definition of J we know that if $J(x) = 0$, then there is a $Q \in M_s(\Omega)$ with its marginal ν at $t = 0$ being a stationary distribution of $\{P^0_\eta, \eta \in E\}$, such that

$$x = \int \omega_0(0)\, Q(d\omega) = \nu(\eta(0) = 1)$$

So $\rho_- \leqslant x \leqslant \rho_+$, i.e., $[m_-, m_+] \subset [\rho_-, \rho_+]$. On the other hand, it is easy to check that for $x \in (\rho_-, \rho_+)$, there exists $p_x > 0$ such that

$$P^0_{\vec{1}}(T_n \geqslant x) \wedge P^0_{-\vec{1}}(T_n \leqslant x) \geqslant p_x$$

Combining this with (5.2) and (5.3) we see that $\rho_- \geqslant m_-$ and $\rho_+ \leqslant m_+$. Thus $m_- = \rho_-$ and $m_+ = \rho_+$, proving the theorem.

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